An Efficient Algorithm for Solving Classes of Integral and Integro-differential Equations

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ABSTRACT
This paper develops a numerical approximation technique based on the structure of differential transform method (DTM), Laplace transform (LT) procedure and Padé approximants (PA) scheme for solving integral and integro-differential equations. The proposed method which gives a virtuous approximation for the true solution in a large region is referred to modified differential transform method (MDTM). Some numerical problems are given to illustrates the efficiency and applicability of the method. Numerical results have shown that the MDTM method is promising for solving integral and integro-differential e quations.

Keywords: Numerical approximation, Differential transform method, Integral equation, Integro differential equations, Volterra integral equations

I. INTRODUCTION

Differential equations play a major role in chemical kinetics, biological models, fluid dynamics, just like in nonlinear equations [17, 18]. Many mathematical formulations of physical phenomena that are modeled under the differential sences usually produce an integro-differential equation (IDE), a differential equation (DE), or likely give an integral equation (IE). However, the integro-differential equation most often contain the other two equations. Given the importance of the integra and integro-differential equations, many studies have been done to find new approximation schemes to these equations or improve the existing methods for their solutions.

Consider the function $u(x)$, let $f(x)$ be a known function, and $K(x,t)$ be the integral kernel. Then, the Volterra integral equation (VIE) of the first kind is defined as:

$$u(x) = \int_a^x K(x,t)u(t)dt, \qquad (1)$$

while the VIE of the second kind is given as:

$$u(x) = \int_a^x K(x,t)u(t)dt, \qquad (2)$$

Solving the Integro-differential equations by analytical procedures for are usually difficult. This and more drawbacks lead to numerous research on efficient approximate solution methods [1]. The DMT method for solving differential equations was introduced by Zhou [2]. This method is an iterative scheme for solving the Taylor series of DE.The DMT method has also been applied to solve initial value and boundary value problems using the concept of Taylor series [1, 3, 4, 16]. The solutions of these problems are usually in series form. The DTM derives an analytical solution in form of a polynomial which are sufficiently differentiable for approximation to exact solutions. However, this method has some setbacks. The DTM gives a truncated series solution which is an accurate approximation of exact solution in a very small region [5].

To improve the precision of differential transform method, we proposed an alternative solution scheme that would modify the series solution for classes integral and integro-differential equations as follows: we begin by applying the Laplace transformation approach to DTM obtained truncated series and use Padé approximants to convert the transformed series into a meromorphic function and further applying the inverse Laplace transform to obtain the desired analytical solution. This would be a better approximation or periodic solution compare to truncated series solution of DTM.

The remaining part of the paper is structured as follow: Section 2 discusses brief overview and preliminary results of DTM, Padé approximants and Laplace transform. Three problems each for both integral and integro-differential equations are presented to illustrates the efficiency and simplicity of the method in section 3. In Section 4, we present the conclusion and discussion for further reference.

II. PRELIMINARIES

This section presents some definitions of differential transform method and Padé approximants.

A. Differential transform method

Definition 1.[6]
Suppose the function $f(x)$ is analytical at x_0 in domain of interest (DOI), then

$$F(k) = \frac{f^{(k)}(x_0)}{k!}. \quad (3)$$

The inverse differential transforms of $F(k)$ is

$$f(x) = \sum_{k=0}^{\infty} F(k)(x - x_0)^k. \quad (4)$$

From (3), (4), we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k. \quad (5)$$

Let $U(k), G(k)$, and $H(k)$ be the differential transforms (DT) of $u(x), g(x)$, and $h(x)$ respectively at $x_0 = 0$. The key operations of the DTM is presented in Table 1.

Table 1: Differential Transform

Original function	Transformed function
$u(x) = g(x) + h(x)$	$U(k) = G(k) + H(k)$
$u(x) = g(x)h(x)$	$U(k) = \sum_{i=1}^{k} G(i)H(k - i)$
$u(x) = cg(x)$	$U(k) = cG(k)$
$u(x) = \frac{d^n g(x)}{dx^n}$	$U(k) = \frac{(k+n)!}{k!}G(k + n)$
$u(x) = x^n$	$U(k) = \delta(k - n)$
$u(x) = \exp(cx)$	$U(k) = \frac{c^k}{k!}$
$u(x) = \cos(\omega x)$	$U(k) = \frac{\omega^k}{k!}\cos\left(\frac{k\pi}{2}\right)$
$u(x) = \sin(\omega x)$	$U(k) = \frac{\omega^k}{k!}\sin\left(\frac{k\pi}{2}\right)$

Theorem 1 [7]
If $f(y) = y^m$, then

$$F(k) = \begin{cases} (Y(0))^m, & k = 0 \\ \frac{1}{Y(0)}\sum_{r=1}^{k}\left(\frac{(m+1)r-k}{k}\right)Y(r)F(k-r), & k \geq 1 \end{cases}$$

Proof. From the definition (1)

$$F(0) = [y^m(x)]_{x=0} = y^m(0) = (Y(0))^m \quad (6)$$

Differentiating $f(y) = y^m$ with respect to x, we get;

$$\frac{df(y)}{dx} = my^{m-1}(x)\frac{dy(x)}{dx} \quad (7)$$

or equivalently,

$$y(x)\frac{df(y(x))}{dx} = mf(y(x))\frac{dy(x)}{dx} \quad (8)$$

By applying the DT to Eq. (8) gives:

$$\sum_{r=0}^{k} Y(r)(k - r + 1)F(k - r + 1) = m\sum_{r=0}^{k}(r + 1)Y(r + 1)F(k - r). \quad (9)$$

From Eq. (9),

$$(k + 1)Y(0)F(k + 1) = m\sum_{r=0}^{k}(r + 1)Y(r + 1)F(k - r) - \sum_{r=1}^{k} Y(r)(k - r + 1)F(k - r + 1). \quad (10)$$

or equivalently,

$$(k + 1)Y(0)F(k + 1) = m\sum_{r=1}^{k+1} rY(r)F(k - r + 1) - \sum_{r=1}^{k} Y(r)(k - r + 1)F(k - r + 1). \quad (11)$$

Thus,

$$(k + 1)Y(0)F(k + 1) = \sum_{r=1}^{k+1}\{(m + 1)r - k - 1\}Y(r)F(k - r + 1). \quad (12)$$

Replacing $k + 1$ by k yields:

$$kY(0)F(k) = \sum_{r=1}^{k}\{(m + 1)r - k\}Y(r)F(k - r). \quad (13)$$

From Eq. (13), if follows that:

$$F(k) = \frac{1}{Y(0)}\sum_{r=1}^{k}\left(\frac{(m+1)r-k}{k}\right)Y(r)F(k - r). \quad (14)$$

Combining Eqs. (6) and (14), we obtain the transformed function of $f(y) = y^m$ as:

$$F(k) = \begin{cases} (Y(0))^m, & k = 0 \\ \frac{1}{Y(0)}\sum_{r=1}^{k}\left(\frac{(m+1)r-k}{k}\right)Y(r)F(k-r), & k \geq 1 \end{cases}$$

This proof shows that other complicated nonlinear functions can be computed in a similar way. Also, nonlinear functions and their transforms are given in the following theorems.

Theorem 2 [7]
If $f(y) = e^{ay}$, then

$$F(k) = \begin{cases} e^{aY(0)}, & k = 0 \\ a\sum_{r=0}^{k-1}\frac{r+1}{k}Y(r + 1)F(k - 1 - r), & k \geq 1 \end{cases}$$

Theorem 3 [7]
If $f(y) = \sin(\alpha y)$ and $g(y) = \cos(\alpha y)$, then

$$F(k) = \begin{cases} \sin(\alpha Y(0)), & k = 0 \\ \alpha \sum_{r=0}^{k-1} \frac{k-r}{k} G(r)Y(k-r), & k \geq 1 \end{cases}$$

and

$$G(k) = \begin{cases} \cos(\alpha Y(0)), & k = 0 \\ -\alpha \sum_{r=0}^{k-1} \frac{k-r}{k} F(r)Y(k-r), & k \geq 1 \end{cases}$$

Theorem 4 [1]
Suppose the DT of the functions $u(x)$ and $g(x)$ are $U(k)$ and $G(k)$, then:

If $f(x) = \int_{x_0}^{x} u(t)dt$, then

$$F(k) = \frac{U(k-1)}{k}, F(0) = 0. \tag{15}$$

If $f(x) = \int_{x_0}^{x} g(t)u(t)dt$, then

$$F(k) = \sum_{l=0}^{k-1} G(l) \frac{U(k-l-1)}{k}, F(0) = 0. \tag{16}$$

If $f(x) = g(x) \int_{x_0}^{x} u(t)dt$, then

$$F(k) = \sum_{l=0}^{k-1} G(l) \frac{U(k-l-1)}{k-l}, F(0) = 0. \tag{17}$$

By means of the DT, the differential equation in DOI is transformed to an algebraic equation in the K-domain and the function $f(t)$ is obtained using the finite-term Taylor series expansion plus a remainder, as

$$f(t) = \sum_{k=0}^{N} F(k) \frac{(t-t_0)^k}{k!} + R_{N+1}(t) \tag{18}$$

In a small region, the series solution in eq. (18) converges rapidly, however, the convergence results are often slow in the wide region and thus, their truncations yield inaccurate results. For further reference on differential transform and DTM see [6, 8-10].

B. Padé approximation [11, 12]
Given the Taylor series expansion of $y(x)$, the Padé approximant is defined as the ratio of two polynomials that derived from the coefficients of $y(x)$. The $[L/M]$ PA to $y(x)$ are defined by

$$\left[\frac{L}{M}\right] = \frac{P_L(x)}{Q_M(x)} \tag{19}$$

where the polynomials $P_L(x)$ and $Q_M(x)$ are of degrees at most L and M. The formal power series

$$y(x) = \sum_{i=1}^{\infty} a_i x^i, \tag{20}$$

$$y(x) - \frac{P_L(x)}{Q_M(x)} = O(x^{L+M+1}) \tag{21}$$

are used to obtain the coefficients of $P_L(x)$ and $Q_M(x)$.

It is obvious that $[L/M]$ will remain unchanged when the denominator and numerator is multiply by certain constant, thus the normalization condition

$$Q_M(0) = 1 \tag{22}$$

is impose on the equation and $P_L(x)$ and $Q_M(x)$ are said to have no common factors. Rewriting the coefficients of the functions $P_L(x)$ and $Q_M(x)$ as

$$\begin{cases} P_L(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_L x^L \\ Q_M(x) = q_0 + q_1 x + q_2 x^2 + \cdots + q_M x^M \end{cases} \tag{23}$$

then, by (22) and (23), the coefficient equations are linearized after multiplying (21) by $Q_M(x)$. Eq (21) is presented as

$$\begin{cases} a_{L+1} + a_L q_1 + \cdots + a_{L-M+1} q_M = 0 \\ a_{L+2} + a_{L+1} q_1 + \cdots + a_{L-M+2} q_M = 0 \\ \vdots \\ a_{L+M} + a_{L+M-1} q_1 + \cdots + a_L q_M = 0 \end{cases} \tag{24}$$

$$\begin{cases} a_0 = p_0 \\ a_0 + a_0 q_1 = p_1 \\ a_2 + a_1 q_1 + a_0 q_2 = p_2 \\ \vdots \\ a_L + a_{L-1} q_1 + \cdots + a_0 q_L = p_L \end{cases} \tag{25}$$

Equation (24) is a set of linear equation. To obtain the solution of this equation, we solve for all the unknown $q's$. Once the values of all the $q's$ are known, then equation (25) would inevitably give the explicit formula for the unknown $p's$, and solution is complete.

However, in the case of non-singularity of equations (24) and (25) would be directly solved to get (26) [12].

$$\left[\frac{L}{M}\right] = \frac{\det \begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ \sum_{j=M}^{L} a_{j-M} x^j & \sum_{j=M-1}^{L} a_{j-M+1} x^j & \cdots & \sum_{j=0}^{L} a_j x^j \end{bmatrix}}{\det \begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ x^M & x^{M-1} & \cdots & 1 \end{bmatrix}} \tag{26}$$

In the case where the lower index on a sum exceed the upper and eq. (26) holds, then, we replaced the sum with zero. The symbolic calculus software, MATLAB is used to obtain diagonal elements of Padé approximants of various orders including: $[2/2]$, $[4/4]$ or $[6/6]$.

Generally, Padé approximant, obtained from a partial Taylor sum is more accurate than the latter. However; Padé, being a rational expression, has poles, which are not present in the original function. It is a simple algebraic task to expand the form of an $[N, M]$ Padé in Taylor series and compute the Padé coefficients by matching with the above [11].

III. NUMERICAL RESULTS

Problem 1: [1]. Consider the linear Volterra integral equation

$$u(t) = 1 - t - \frac{t^2}{2} + \int_0^t (t-x)u(x)dx, 0 < x < 1. \quad (27)$$

and exact solution $u(t) = 1 - \sinh(t)$. Clearly, $u(0) = 1$, from Theorem 4 and using operations from Table 1, we have:

$$U(k) = \delta(k) - \delta(k-1) - \frac{\delta(k-2)}{2} + \sum_{l=0}^{k-1} \delta(l-1)\frac{U(k-l-1)}{k-l} - \sum_{l=0}^{k-1} \delta(l-1)\frac{U(k-l-1)}{k}, k \geq 1,$$

transforming the initial condition by (3) to get

$$U(0) = 1.$$

Subsequently, we find

$$U(1) = -1, U(2) = 0, U(3) = -\frac{1}{6}, U(4) = 0, \ U(5) = -\frac{1}{120}, U(6) = 0, U(7) = -\frac{1}{5040}.$$

Applying the inverse transformation rule of eq. (4), the approximate solution of eq. (27) is given as follows:

$$u(t) = \sum_{k=0}^{\infty} U(k)t^k = 1 - t - \frac{t^3}{6} - \frac{t^5}{120} - \frac{t^7}{5040} + \cdots \quad (28)$$

which in the limit of infinitely many terms yields the exact solution of eq. (27).

To improve the accuracy of (28), the proposed MDTM was implemented on first three terms of (28) as follows:

Applying the Laplace transform to the first three terms from (28), yields

$$\mathcal{L}(y(t)) = \frac{1}{s} - \frac{1}{s^2} - \frac{1}{s^4}.$$

For simplicity, let $s = \frac{1}{z}$; then

$$\mathcal{L}(y(t)) = z - z^2 - z^4. \quad (29)$$

The Padé approximants $\left[\frac{2}{2}\right]$ gives

$$\left[\frac{2}{2}\right] = \frac{z^3 + z^2 - z}{z^2 - 1}$$

Recalling $z = \frac{1}{s}$, we obtain $\left[\frac{2}{2}\right]$ in terms of s

$$\left[\frac{2}{2}\right] = \frac{-s^2 + s + 1}{-s^3 + s}$$

Applying the inverse Laplace transform (ILT) to Padé approximant of order $\left[\frac{2}{2}\right]$ gives the modified approximate solution:

$$u(t) = -\frac{e^t}{2} + \frac{e^{-t}}{2} + 1 = 1 - \sinh(t).$$

Problem 2: [1]. Consider the nonlinear Volterra integral equation

$$y(t) + \int_0^t \left(y(x) + y^2(x)\right)dx = \frac{3}{2} - \frac{1}{2}e^{-2t} \quad (30)$$

and exact solution

$$y(t) = e^{-t}.$$

It is obvious that $y(0) = 1$, then, from Theorem 4 and Table 1, the recurrence relation follows:

$$Y(k) + \frac{Y(k-1)}{k} + \sum_{l=0}^{k-1} Y(l)\frac{Y(k-l-1)}{k-l} = \frac{3}{2}\delta(k) - \frac{1}{2}\frac{(-2)^k}{k!}, k \geq 1 \quad (31)$$

transforming the initial condition by (3) to get $Y(0) = 1$.

Consequently, we find

$$Y(1) = -1, Y(2) = \frac{1}{2}, Y(3) = -\frac{1}{6}, Y(4) = \frac{1}{24}, Y(5) = -\frac{1}{120}, Y(6) = \frac{1}{720}, Y(7) = -\frac{1}{5040}.$$

Under the inverse transformation rule (4), the following approximate solution of equation (30) is obtained

$$y(t) = \sum_{k=0}^{\infty} Y(k)t^k = 1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} - \frac{t^5}{120} + \frac{t^6}{720} - \frac{t^7}{5040} + \cdots, k \geq 1 \quad (32)$$

which yields the exact solution of eq. (30) in limit of infinitely many terms. Next, we implement MDTM on first four term of (32) as follows:

Applying the LT to the first four terms from (32), yields

$$\mathcal{L}(y(t)) = \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s^4}.$$

For simplicity, let $s = \frac{1}{z}$; then

$$\mathcal{L}(y(t)) = z - z^2 + z^3 - z^4., k \geq 1 \quad (33)$$

The Padé approximants $\left[\frac{2}{2}\right]$ gives

$$\left[\frac{2}{2}\right] = \frac{z}{z+1}.$$

Recalling $z = \frac{1}{s}$, we obtain $\left[\frac{2}{2}\right]$ in terms of s

$$\left[\frac{2}{2}\right] = \frac{1}{s+1}$$

Applying the ILT to $\left[\frac{2}{2}\right]$ Padé approximant gives the modified approximate solution

$$y(t) = e^{-t}.$$

Problem 3: [1]. Given the following nonlinear Volterra integral equation

$$y(t) = \cos(t) + \frac{1}{2}\sin(2t) + 3t - 2\int_0^t \left(1 + y^2(x)\right)dx \qquad (34)$$

whose exact solution is $y(t) = \cos(t)$.
By simplifying (34), to get

$$y(t) = \cos(t) + \frac{1}{2}\sin(2t) + 3t - 2t - \int_0^t y^2(x)dx \qquad (35)$$

So,

$$y(t) = \cos(t) + \frac{1}{2}\sin(2t) + t - \int_0^t y^2(x)dx \qquad (36)$$

Clearly, $y(0) = 1$, then, from Theorem 4 and operations in Table 1, we obtained the recurrence relation:

$$Y(k) = \frac{1}{k!}\cos\left(\frac{\pi k}{2}\right) + \frac{2^{(k-1)}}{k!}\sin\left(\frac{\pi k}{2}\right) + \delta(k-1) - \sum_{l=0}^{k-1} Y(l)\frac{Y(k-l-1)}{k-l},\ k \geq 1 \qquad (37)$$

transforming the initial condition by (3) to get

$$Y(0) = 1.$$

Consequently, we find

$$Y(1) = 0, Y(2) = -\frac{1}{2}, Y(3) = 0, Y(4) = \frac{1}{4!}, Y(5) = 0, Y(6) = -\frac{1}{6!}, Y(7) = 0, Y(8) = \frac{1}{8!}.$$

The approximate solution of eq. (34) is obtained using the inverse transformation rule (4) as follows:

$$y(t) = \sum_{k=0}^{\infty} Y(k)t^k = 1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} + \cdots \qquad (38)$$

which yields the exact solution of (34) in the limit of infinitely many terms. Taking just the first two terms from (38), we implement the MDTM as follows:

Applying the LT procedure to the first two terms from (38), yields

$$\mathcal{L}(y(t)) = \frac{1}{s} - \frac{1}{s^3}.$$

For simplicity, let $s = \frac{1}{z}$; then

$$\mathcal{L}(y(t)) = z - z^3. \qquad (39)$$

The Padé approximants $\left[\frac{2}{2}\right]$ gives

$$\left[\frac{2}{2}\right] = \frac{z}{z^2+1}.$$

Recalling $z = \frac{1}{s}$, we obtain $\left[\frac{2}{2}\right]$ in terms of s

$$\left[\frac{2}{2}\right] = \frac{s}{s^2+1}.$$

The modified approximate solution $y(t) = \cos(t)$. is obtained using the inverse Laplace transform to the $\left[\frac{2}{2}\right]$

Problem 4: [13]. Given the following nonlinear Volterra integro–differential equation

$$y'(t) = \frac{3}{2}e^t - \frac{1}{2}e^{3t} + \int_0^t e^{t-x}y^3(x)dx,\quad y(0) = 1. \qquad (40)$$

whose exact solution is $y(t) = e^t$. By simplifying (40), to get

$$y'(t) = \frac{3}{2}e^t - \frac{1}{2}e^{3t} + \int_0^t e^{t-x}y^3(x)dx,\quad y(0) = 1. \qquad (41)$$

Using operations in Table 1 and Theorem 4, we obtain the recurrence relation as follows:

$$Y(k+1) = \frac{1}{k+1}\left[\frac{3}{2(k!)} - \frac{3^k}{2(k!)} + \sum_{l=0}^{k-1}\frac{1}{l!}\frac{G(k-l-1)}{k-l}\right],\quad k \geq 1, \qquad (42)$$

where $G(k)$ is the differential transform for $g(x) = e^{-x}y^3(x)$

$$G(k) = \sum_{i=0}^{k}\frac{(-1)^i}{i!}H(k-i) \qquad (43)$$

where $H(k)$ is the differential transform for $h(x) = y^3(x)$

$$H(k) = \begin{cases} 1, & k = 0 \\ \sum_{r=1}^{k}\left(\frac{4r-k}{k}\right)Y(r)F(k-r), & k \geq 1 \end{cases} \qquad (44)$$

transforming the initial condition by (3) to get $Y(0) = 1$. Then substituting in (44) and (43), respectively, to get $H(0) = 1, G(0) = 1$.
Consequently, we find

$$Y(1) = 1, Y(2) = \frac{1}{2}, Y(3) = \frac{1}{3!}, Y(4) = \frac{1}{4!}, Y(5) = \frac{1}{5!}, Y(6) = \frac{1}{6!}, Y(7) = \frac{1}{7!}.$$

The approximate solution of eq. (40) is obtained using the inverse transformation rule (4) as follows:

$$y(t) = \sum_{k=0}^{\infty} Y(k)t^k = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!} + \cdots \qquad (45)$$

which yields the exact solution of (40) in the limit of infinitely many terms. Taking the first three terms from (45), we implement the MDTM as follows:

Applying the LT to the first three terms from (45), yields

$$\mathcal{L}(y(t)) = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3}.$$

For simplicity, let $s = \frac{1}{z}$; then

$$\mathcal{L}(y(t)) = z + z^2 + z^3. \quad (46)$$

The Padé approximants $\left[\frac{2}{2}\right]$ gives

$$\left[\frac{2}{2}\right] = -\frac{z}{z-1}.$$

Recalling $z = \frac{1}{s}$, we obtain $\left[\frac{2}{2}\right]$ in terms of s

$$\left[\frac{2}{2}\right] = \frac{1}{s-1}.$$

The modified approximate solution $y(t) = e^t$ is obtained using inverse Laplace transform to the $\left[\frac{2}{2}\right]$ Padé approximant.

Problem 5: [14]. Consider the following linear boundary value problem for integro-differential equation

$$y^{(4)}(t) = t(1+e^t) + 3e^t + y(t) - \int_0^t y(x)dx, 0 < t < 1 \quad (47)$$

whose boundary conditions is given as

$$y(0) = y'(0) = 1, \quad y(1) = 1 + e, \quad y'(1) = 2e, \quad (48)$$

and exact solution $y(t) = 1 + te^t$.

Transforming Eq. (47) and Eq. (48), we obtain

$$Y(k+4) = \frac{k!}{(k+5)!}\left[\delta(k-1) + \sum_{i=0}^{k}\frac{\delta(i-1)}{(k-i)!} + \frac{3}{k!} + Y(k) - \frac{Y(k-1)}{k}\right] \quad (49)$$

$$Y(0) = 1, Y(1) = 1, \ Y(2) = \frac{A}{2}, \ Y(3) = \frac{B}{3!}. \quad (50)$$

Substitute (50) in (49), to get

$Y(4) = \frac{1}{3!}, Y(5) = \frac{1}{4!}, Y(6) = \frac{(A+4)}{6!}, \ Y(7) = \frac{(B-A+6)}{7!}, Y(8) = \frac{(11-B)}{8!}, \ Y(9) = \frac{1}{8!}, Y(10) = \frac{(A+8)}{10!}.$

The approximate solution of equation (47) is obtained using the inverse transformation rule (4) as follows:

$$y(t) = \sum_{k=0}^{\infty} Y(k)t^k$$
$$= 1 + t + \frac{A}{2}t^2 + \frac{B}{3!}t^3 + \frac{1}{3!}t^4 + \frac{1}{4!}t^5 + \frac{(A+4)}{6!}t^6 + \frac{(B-A+6)}{7!}t^7 + \frac{(11-B)}{8!}t^8 + \frac{1}{8!}t^9 + \frac{(A+8)}{10!}t^{10} + \ldots \quad (51)$$

To find the values of constants A and B, we will substitute the initial conditions $y(1) = 1 + e, \ y'(1) = 2e$ in Eq. (51), to get $y(1) = 1 + e$, then

$$1 + 1 + \frac{A}{2} + \frac{B}{3!} + \frac{1}{3!} + \frac{1}{4!} + \frac{(A+4)}{6!} + \frac{(B-A+6)}{7!} + \frac{(11-B)}{8!} + \frac{1}{8!} + \frac{(A+8)}{10!} = 1 + e,$$

that gives

$$\frac{1818721}{3628800}A + \frac{961}{5760}B = -\frac{17228}{14175} + e \quad (52)$$

and

$$y' = 1 + At + \frac{B}{2}t^2 + \frac{2}{3}t^3 + \frac{5}{24}t^4 + \frac{(A+4)}{5!}t^5 + \frac{(B-A+6)}{6!}t^6 + \frac{(11-B)}{7!}t^7 + \frac{9}{8!}t^8 + \frac{(A+8)}{9!}t^9 + \cdots,$$

and $y'(1) = 2e$, then

$$\frac{365401}{362880}A + \frac{421}{840}B = -\frac{696401}{362880} + 2e \quad (53)$$

From (52) and (53), $A = 1.999995083333181$, $B = 3.000016585166542$. For approximations of A and B,. It is obvious that

$$\lim_{n\to\infty} A = 2, \quad \lim_{n\to\infty} B = 3. \quad (54)$$

We implement MDTM to the first five terms as follows: Substituting $A = 2, B = 3$ in (51), then applying the Laplace transform yields

$$\mathcal{L}(y(t)) = \frac{1}{s} + \frac{1}{s^2} + \frac{2}{s^3} + \frac{3}{s^4} + \frac{4}{s^5}.$$

For simplicity, let $s = \frac{1}{z}$; then

$$\mathcal{L}(y(t)) = z + z^2 + 2z^3 + 3z^4 + 4z^5, \quad (55)$$

The Padé approximants $\left[\frac{3}{2}\right]$ gives

$$\left[\frac{3}{2}\right] = \frac{z^3 - z^2 + z}{z^2 - 2z + 1}.$$

Recalling $z = \frac{1}{s}$, we obtain $\left[\frac{3}{2}\right]$ in terms of s

$$\left[\frac{3}{2}\right] = \frac{s^2 - s + 1}{s^3 - 2s^2 + s}.$$

The modified approximate solution $y(t) = te^t + 1$. is obtained using the inverse Laplace transform to the $\left[\frac{3}{2}\right]$ Padé approximant.

Problem 6: [15]. Consider the following nonlinear boundary value problem for integro-differential equation

$$y^{(4)}(t) = 1 + \int_0^t e^{-x} y^2(x)dx, 0 < t < 1 \quad (56)$$

whose boundary conditions

$$y(0) = y'(0) = 1, y(1) = y'(1) = e, \quad (57)$$

and exact solution $y(t) = e^t$. Transforming Eq. (56) and Eq. (57), we obtain

$$Y(k+4) = \frac{k!}{(k+4)!}\left[\delta(k) + \sum_{i=0}^{k-1} \frac{(-1)^i}{i!} \frac{G(k-i-1)}{k}\right]. \quad (58)$$

where $G(k)$ is the differential transform of $g(y) = y^2$.

$$Y(0) = 1, Y(1) = 1, \ Y(2) = \frac{A}{2}, \ Y(3) = \frac{B}{3!}. \quad (59)$$

By Theorem 1, the differential transform $G(k)$ in equation (3.32) is

$$G(0) = \left(Y(0)\right)^2 = 1, \quad (60)$$

$$G(k) = \sum_{r=1}^{k} \left(\frac{3r-k}{k}\right) Y(r)G(k-r), \qquad k \geq 1. \quad (61)$$

Therefore, $G(1) = 2$, then

$$Y(4) = \frac{1}{4!}, Y(5) = \frac{1}{5!}, Y(6) = \frac{1}{6!}.$$

The approximate solution of equation (56) is obtained using the inverse transformation rule (4) as follows:

$$y(t) = \sum_{k=0}^{\infty} Y(k)t^k \quad (62)$$

$$= 1 + t + \frac{A}{2}t^2 + \frac{B}{3!}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5 + \frac{1}{6!}t^6 + \frac{(A-3)}{7!}t^7 + \cdots$$

Now, to find the values of constants A and B, we will substitute the initial conditions $y(1) = y'(1) = e$, in Eq. (62), to get $y(1) = e$, then

$$1 + 1 + \frac{A}{2} + \frac{B}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{A-3}{5040} = e,$$

then

$$\frac{A}{2} + \frac{B}{6} = e - \frac{646}{315} \quad (63)$$

And $y' = 1 + At + \frac{B}{2}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5 + \cdots$, $y'(1) = e$, then $1 + A + \frac{B}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{A-3}{720} = e$, thus

$$A + \frac{B}{2} = e - \frac{97}{80} \quad (64)$$

From (63) and (64),
$A = 0.993365409074278, B = 1.024832838769536$. For approximations of A and B, It is obvious that

$$\lim_{n\to\infty} A = 1, \qquad \lim_{n\to\infty} B = 1.$$

Substituting $A = 1, B = 1$ in (62) yields the exact solution of (56) in the limit of infinitely many terms. Taking just the first three terms from (56), the MDTM is implemented as follows:

Applying the Laplace transform to the first three terms from the series solution (56), yields

$$\mathcal{L}\left(y(t)\right) = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3}. \quad (65)$$

For simplicity, let $s = \frac{1}{z}$; then

$$\mathcal{L}\left(y(t)\right) = z + z^2 + z^3. \quad (66)$$

The Padé approximants $\left[\frac{2}{2}\right]$ gives

$$\left[\frac{2}{2}\right] = -\frac{z}{z-1}.$$

Recalling $z = \frac{1}{s}$, we obtain $\left[\frac{2}{2}\right]$ in terms of s

$$\left[\frac{2}{2}\right] = \frac{1}{s-1}.$$

The approximate solution $y(t) = e^t$. is obtained by inverse Laplace transform to the $\left[\frac{2}{2}\right]$ Padé approximant.

IV. CONCLUSION AND DISCUSSION

The MDTM is an efficient scheme for approximating solutions of integral and integro-differential equations. This method employs the Laplace transformation technique to the truncated series obtained by DTM, the Padé approximants for converting the transformed series into meromorphic function before applying the Laplace transform inverse to solve the analytical problem. All results obtained show that MDTM is in good agreement the results obtained by the exact solution. Examples presented indicates that MDTM greatly improves the convergence rate of DTM's truncated series solution with true analytic solution. The results also show that this scheme is promising and can be applied in other applications.

Acknowledgments

The authors are grateful to the government of Malaysia for funding this research and thankful to the reviewers for their valuable comments and suggestions.

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